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Group-Valued Measures on the Lattice of Closed Subspaces of a Hilbert Space

John Harding,[1,3] Ekaterina Jager,[2] and Derek Smith[2]

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We show there are no non-trivial finite Abelian group-valued measures on the lattice of closed subspaces of an infinite-dimensional Hilbert space, and we use this to establish that the unigroup of the lattice of closed subspaces of an infinite-dimensional Hilbert space is divisible. The main technique is a combinatorial construction of a set of vectors in $\mathbb{R}^{2n}$ generalizing properties of those used in various treatments of the Kochen–Specker theorem in $\mathbb{R}^4$.

KEY WORDS: group-valued measure; Hilbert space; Kochen–Specker theorem; unigroup.

1. INTRODUCTION

A group-valued measure on an orthomodular lattice L is a map $f : L \to G$, of L into an Abelian group G, that satisfies $f(x \vee y) = f(x) + f(y)$ for all $x, y \in L$ with $x \leq y'$. There have been numerous studies of group-valued measures on Boolean algebras, orthomodular lattices, and their generalizations (Avallone and Hamhalter, 1996; D'Andrea and De Lucia, 1992; De Lucia and Morales, 1998; García Mazarío, 2001; Göbel *et al.*, 1996; Navara *et al.*, 1993; Bhashkara Rao and Shortt, 1991). Additionally, the group-valued measures on an orthomodular structure L are closely linked to the unigroup of L, a key ingredient in David Foulis' recent algebraic approach to non-commutative measure theory (Bennett and Foulis, 1997; Foulis *et al.*, 1998; Foulis, 2000).

Our purpose here is to study the group-valued measures on the orthomodular lattices $L(\mathcal{H})$ of closed subspaces of a Hilbert space $\mathcal{H}$ (Kalmbach, 1983). Of course, a great deal is known about the $\mathbb{R}$-valued measures on $L(\mathcal{H})$ (Dvurečenskij, 1993), but little is known about measures from $L(\mathcal{H})$ into other groups. For

[1] Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico.
[2] Department of Mathematics, Lafayette College, Easton, Pennsylvania.
[3] To whom correspondence should be addressed at Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003; e-mail: jharding@nmsu.edu.

instance, it seems a completely open problem whether there are any $\mathbb{Z}_2$-valued measures on $L(\mathbb{R}^3)$ beside the two obvious ones.

The only real progress on the type of question we ask was made by Navara and Pták in (Navara and Pták, 2004) where they showed that any $\mathbb{Z}_2$-valued measure on $L(\mathbb{R}^5)$ is constant on the atoms (one-dimensional subspaces) of $L(\mathbb{R}^5)$. It follows that there are exactly two $\mathbb{Z}_2$-valued measures on $L(\mathbb{R}^5)$—one that is identically zero, and another that is given by taking the dimension of a subspace modulo 2. Navara and Pták obtained their result by using a configuration of vectors in $\mathbb{R}^4$ similar to one used by Peres in his treatment of the Kochen–Specker theorem (Peres, 1993).

The key ingredient in this paper is a combinatorial construction of a set of vectors in $\mathbb{R}^{2n}$, for any $n \geq 2$, that generalizes to higher dimensions properties of sets of vectors used by Peres and others (Cabello *et al.*, 1999; Kernaghan, 1994; Mermin *et al.*, 1990; Peres, 1993; Smith, 2004) in discussions of the Kochen–Specker theorem in $\mathbb{R}^4$. Using these vectors we are able to show that if $m \geq 2n + 1$, then all $\mathbb{Z}_n$-valued measures on $L(\mathbb{R}^m)$ are built in an obvious way from the dimension function on $L(\mathbb{R}^m)$. Indeed, we strengthen this result somewhat to provide such a characterization of $\mathbb{Z}_{n^k}$-valued measures on $L(\mathbb{R}^m)$ whenever $k \geq 1$ and $m \geq 2n + 1$.

These results are then used to show that for $\mathcal{H}$ an infinite-dimensional Hilbert space and G a finite Abelian group, all G-valued measures on $L(\mathcal{H})$ are constantly zero. While this is far from a characterization of the group-valued measures on $L(\mathcal{H})$, it does provide sufficient information for us to conclude that for $\mathcal{H}$ an infinite-dimensional Hilbert space, the unigroup of $L(\mathcal{H})$ is a divisible group.

While all our results carry over directly to complex Hilbert space setting, for economy of presentation, we state our results only for real Hilbert spaces.

2. A CONSTRUCTION OF A SET OF VECTORS IN $\mathbb{R}^{2N}$

We first introduce some terminology. Suppose V is an n-dimensional vector space over the reals. A *block* B of vectors in V is a set of n pairwise orthogonal non-zero vectors in V. A *k-simplex* S in V is a set of $k + 1$ non-zero vectors in V such that (i) all vectors in S have the same length, and (ii) any two distinct vectors in S have the same inner product. The key technical result of this paper, Lemma 1, uses two orthogonal n-simplices to construct blocks in $\mathbb{R}^{2n}$ with certain intersection properties.

Lemma 1. *For any $n \geq 2$ there is a set Z of vectors in $\mathbb{R}^{2n}$ such that*

1. *Z has $2(n + 1)^2$ vectors.*
2. *Z can be covered by a family of $(n + 1)^2$ blocks B_{ij} where $0 \leq i, j \leq n$.*
3. *Each vector in Z occurs in exactly n of the blocks B_{ij} where $0 \leq i, j \leq n$.*

Proof: Let $e_0, \dots, e_n, f_0, \dots, f_n$ be an orthonormal basis of $\mathbb{R}^{2n+2}$ and set

$$e = e_0 + \cdots + e_n \quad \text{and} \quad f = f_0 + \cdots + f_n.$$

We consider the $2n$-dimensional subspace W of vectors orthogonal to both e and f. Note that as W is isomorphic to $\mathbb{R}^{2n}$, it is enough to establish our result in W.

For each $0 \le i \le n$ define vectors

$$s_i = e - (n+1)e_i \quad \text{and} \quad t_i = f - (n+1)f_i.$$

We then set $S = \{s_0, \dots, s_n\}$ and $T = \{t_0, \dots, t_n\}$.

We claim that S and T are orthogonal n-simplices in W. Indeed, as $\|e\|^2 = n+1$, the inner product $e \cdot s_i = \|e\|^2 - (n+1)(e \cdot e_i) = 0$. It follows that each vector in S, and by symmetry each vector in T, is orthogonal to both e and f. Thus S and T are subsets of W. The symmetry in the definition of these vectors yields that S and T are n-simplices in W and clearly each vector in S is orthogonal to each vector in T.

We use the simplices S, T to build a set Z of vectors in W and blocks B_{ij} for $0 \le i, j \le n$ that cover Z in the desired manner. We first set

$$r = n^{-1/2}.$$

Then for each $0 \le i, j \le n$ define

$$u_{ij} = rs_i + t_j \quad \text{and} \quad v_{ij} = s_i - rt_j.$$

We then set

$$Z = \{u_{ij}, v_{ij} : 0 \le i, j \le n\},$$

and for $0 \le i, j \le n$ we define

$$B_{ij} = \{u_{ik} : k \ne j\} \cup \{v_{kj} : k \ne i\}.$$

Note that Z is a collection of $2(n+1)^2$ vectors in W, that each B_{ij} is a collection of $2n$ vectors in W and that each vector in Z occurs in exactly n of the sets B_{ij}. It remains only to show that the B_{ij} are blocks of W, or equivalently, that the vectors in each B_{ij} are pairwise orthogonal.

For $0 \le i \le n$, as $s_i = e - (n+1)e_i = e_0 + \cdots - ne_i + \cdots + e_n$ we have

$$\|s_i\|^2 = n^2 + n, \quad \text{and similarly} \quad \|t_i\|^2 = n^2 + n.$$

And for $0 \le i \ne j \le n$ we have

$$s_i \cdot s_j = -(n+1), \quad \text{and similarly} \quad t_i \cdot t_j = -(n+1).$$

Therefore, for $k \ne m$ we have $t_k \cdot t_m = -r^2\|s_i\|^2$ for each $0 \le i \le n$, giving

$$u_{ik} \cdot u_{im} = (rs_i + t_k) \cdot (rs_i + t_m) = r^2\|s_i\|^2 + 0 + 0 - r^2\|s_i\|^2 = 0.$$

Similarly, for $k \neq m$ and $0 \leq j \leq n$, we have

$$v_{kj} \cdot v_{mj} = 0.$$

Finally, if $k \neq j$ and $m \neq i$, then $r(s_i \cdot s_m) = r(t_k \cdot t_j)$ for each $0 \leq i, j \leq n$, giving

$$u_{ik} \cdot v_{mj} = (rs_i + t_k) \cdot (s_m - rt_j) = r(s_i \cdot s_m) - 0 + 0 - r(t_k \cdot t_j) = 0.$$

Thus, the vectors in B_{ij} are orthogonal, showing each B_{ij} is a block. $\square$

3. RESULTS IN FINITE DIMENSIONS

We next use the vectors created in the previous section to obtain results about finite group-valued measures on the lattice of closed subspaces of $\mathbb{R}^n$. By way of notation, we use $\langle v \rangle$ for the one-dimensional subspace spanned by a vector v, and if S, T are orthogonal subspaces of $\mathbb{R}^n$ we use $S \oplus T$ for the subspace spanned by S and T. Recall that any group-valued measure f satisfies $f(S \oplus T) = f(S) + f(T)$. We use $b \pmod n$ for the unique natural number a with $a < n$ and $b = kn + a$ for some integer k, and we let $\mathbb{Z}_n$ be the finite cyclic group $\{0, \ldots, n-1\}$ equipped with the operation of addition modulo n.

Lemma 2. *Any $\mathbb{Z}_n$-valued measure f on $L(\mathbb{R}^{2n})$ satisfies $f(\mathbb{R}^{2n}) = 0$.*

Proof: Use Lemma 1 to obtain a set Z of vectors and a family B_{ij} for $0 \leq i, j \leq n$ of blocks in $\mathbb{R}^{2n}$ such that each vector in Z occurs in exactly n of the blocks B_{ij}. Enumerate the vectors in each block B_{ij} as v_{ijk} where $1 \leq k \leq 2n$. We then have

$$\begin{aligned}
\sum_{ijk} f(\langle v_{ijk} \rangle) &= \sum_{ij} (f(\langle v_{ij1} \rangle) + \cdots + f(\langle v_{ij\,2n} \rangle)) \\
&= \sum_{ij} f(\langle v_{ij1} \rangle \oplus \cdots \oplus \langle v_{ij\,2n} \rangle) \\
&= \sum_{ij} f(\mathbb{R}^{2n}) \\
&= (n+1)^2 f(\mathbb{R}^{2n}) \pmod n \\
&= f(\mathbb{R}^{2n}).
\end{aligned}$$

As each vector in Z occurs n times in the list v_{ijk} where $0 \leq i, j \leq n$, $1 \leq k \leq 2n$,

$$\sum_{ijk} f(\langle v_{ijk} \rangle) = n \sum_{z \in Z} f(\langle z \rangle) \pmod n = 0.$$

This establishes the result. $\square$

Theorem 3. *If $m \geq 2n+1$, the $\mathbb{Z}_n$-valued measures on $L(\mathbb{R}^m)$ are exactly the maps*

$$f(A) = k \cdot \dim(A) \pmod{n}$$

for some $0 \leq k < n$.

Proof: For orthogonal subspaces A and B we have $\dim(A \oplus B) = \dim(A) + \dim(B)$, therefore each of the indicated maps is a Z_n-valued measure on $L(\mathbb{R}^m)$.

Suppose that f is any $\mathbb{Z}_n$-valued measure on $L(\mathbb{R}^m)$ and let u and v be any non-zero vectors in $\mathbb{R}^m$. As $m \geq 2n+1$ there is a $2n+1$-dimensional subspace S of $\mathbb{R}^m$ that contains u and v. By $\langle u \rangle^\perp$ and $\langle v \rangle^\perp$ we shall mean the subspaces of S orthogonal to u and v respectively. As $\langle u \rangle \oplus \langle u \rangle^\perp = S$ and $\langle v \rangle \oplus \langle v \rangle^\perp = S$,

$$f(\langle u \rangle) + f(\langle u \rangle^\perp) = f(\langle v \rangle) + f(\langle v \rangle^\perp).$$

Note, if T is a subspace of $\mathbb{R}^m$ then f restricts to a $\mathbb{Z}_n$-valued measure on $L(T)$. In particular, f restricts to a $\mathbb{Z}_n$-valued measure on $L(\langle u \rangle^\perp)$ and on $L(\langle v \rangle^\perp)$. Since $\langle u \rangle^\perp$ and $\langle v \rangle^\perp$ are $2n$-dimensional, it then follows from Lemma 2 that $f(\langle u \rangle^\perp) = 0$ and $f(\langle v \rangle^\perp) = 0$. Applying this to the above equation then yields that $f(\langle u \rangle) = f(\langle v \rangle)$.

We have shown that f takes a constant value, say k, on all one-dimensional subspaces $\langle v \rangle$. It follows by additivity that $f(A) = k \cdot \dim(A) \pmod{n}$ for all subspaces A. □

Corollary 4. *If $m \geq 2n+1$, then for any $p \geq 1$ the $\mathbb{Z}_{n^p}$-valued measures on $L(\mathbb{R}^m)$ are exactly the maps*

$$f(A) = k \cdot \dim(A) \pmod{n^p}$$

for some $0 \leq k < n^p$.

Proof: The proof is by induction on p. If $p = 1$ the result is given by Theorem 3. Suppose then that $p > 1$ and f is a $\mathbb{Z}_{n^p}$-valued measure on $L(\mathbb{R}^m)$.

Consider the measure $g : L(\mathbb{R}^m) \to \mathbb{Z}_{n^p}$ defined by

$$g(A) = n \cdot f(A) \pmod{n^p}.$$

Note that the image of g is contained in the subgroup $n\mathbb{Z}_{n^p}$ consisting of all elements of $\mathbb{Z}_{n^p}$ that are multiples of n. As $n\mathbb{Z}_{n^p}$ is isomorphic to $\mathbb{Z}_{n^{p-1}}$, it follows from the inductive hypothesis that g takes a constant value on the one-dimensional subspaces of $\mathbb{R}^m$. Thus there is some $b \in n\mathbb{Z}_{n^p}$ so that for all one-dimensional subspaces $\langle v \rangle$

$$g(\langle v \rangle) = b.$$

As $b \in n\mathbb{Z}_{n^p}$ there is some $r \in \mathbb{Z}_{n^p}$ with $b = nr$. We claim that for any one-dimensional subspace $\langle v \rangle$,

$$r = f(\langle v \rangle) \pmod{n^{p-1}}.$$

Suppose $f(\langle v \rangle) = r'$. Then by the definition of g and the fact that $g(\langle v \rangle) = b$ we have $b = nr' \pmod{n^p}$. Thus $0 = n(r - r') \pmod{n^p}$, giving $r = r' \pmod{n^{p-1}}$.

Consider the measure $h : L(\mathbb{R}^m) \to \mathbb{Z}_{n^p}$ defined by

$$h(A) = r \cdot \dim(A) \pmod{n^p}.$$

Then $f - h$ is a $\mathbb{Z}_{n^p}$-valued measure on $L(\mathbb{R}^m)$. Note that for any one-dimensional subspace $\langle v \rangle$ we have $h(\langle v \rangle) = r$, and therefore

$$0 = (f - h)(\langle v \rangle) \pmod{n^{p-1}}.$$

Thus the image of $f - h$ is contained in the subgroup $n^{p-1}\mathbb{Z}_{n^p}$ of $\mathbb{Z}_{n^p}$. As $n^{p-1}\mathbb{Z}_{n^p}$ is isomorphic to $\mathbb{Z}_n$, the inductive hypothesis (or Theorem 3) yields that $f - h$ is constant on all the one-dimensional subspaces $\langle v \rangle$. Then as $f - h$ and h are both constant on the one-dimensional subspaces, f is also constant on the one-dimensional subspaces. The result follows. $\square$

We are not aware of any versions of Theorem 3 or Corollary 4 with $m \leq 2n$.

4. RESULTS FOR INFINITE-DIMENSIONAL HILBERT SPACES

We next apply our results to the infinite-dimensional setting. The reader should consult (Kadison and Ringrose, 1983, pp. 88–94) for background on infinite-dimensional Hilbert spaces.

Lemma 5. *Suppose $\mathcal{H}$ is an infinite-dimensional Hilbert space. Then for any natural number $k > 0$ there is an ortholattice embedding of $L(\mathbb{R}^k)$ into $L(\mathcal{H})$.*

Proof: The following is a glorified version of the embedding of $L(\mathbb{R}^2)$ into $L(\mathbb{R}^4)$ that sends a subspace A of $\mathbb{R}^2$ to the subspace of $\mathbb{R}^4$ consisting of all vectors (c_0, c_1, c_2, c_3) for which both (c_0, c_1) and (c_2, c_3) belong to A.

Note that each ordinal α can be uniquely expressed as $\alpha = \beta + n$ where β is a limit ordinal and n is a natural number. If $\alpha = \beta + n$ is such a representation, we say $0 = \alpha (\operatorname{mod} k)$ if $0 = n (\operatorname{mod} k)$.

Suppose $\mathcal{H}$ is of dimension κ and $\{v_\alpha : \alpha < \kappa\}$ is an orthonormal basis of $\mathcal{H}$. Each $v \in \mathcal{H}$ can be uniquely expressed as a sum $v = \sum\{c_\alpha v_\alpha : \alpha < \kappa\}$ for some choice of scalars c_α (with all but countably many being zero). For such v we define for each $\alpha < \kappa$ with $0 = \alpha \pmod{k}$ a vector $v^\alpha \in \mathbb{R}^k$ by setting

$$v^\alpha = (c_\alpha, \ldots, c_{\alpha+k-1}).$$

We then define a map Γ from $L(\mathbb{R}^k)$ to $L(\mathcal{H})$ by setting

$$\Gamma(A) = \{v \in \mathcal{H} : v^\alpha \in A \text{ for all } \alpha < \kappa \text{ with } 0 = \alpha \pmod k\}.$$

It is routine to verify that Γ is an ortholattice embedding. □

Theorem 6. *If $\mathcal{H}$ is an infinite-dimensional Hilbert space, then any group-valued measure $f : L(\mathcal{H}) \to G$ into a finite Abelian group G identically zero.*

Proof: We first show any group-valued measure $g : L(\mathcal{H}) \to \mathbb{Z}_n$ into a finite cyclic group is identically zero. Suppose S is an infinite-dimensional closed subspace of $\mathcal{H}$. Then $L(S)$ is the interval $[\{0\}, S]$ in the orthomodular lattice $L(\mathcal{H})$, and therefore g restricts to a group-valued measure from $L(S)$ to $\mathbb{Z}_n$. By Lemma 5, there is an ortholattice embedding i of $L(\mathbb{R}^{2n})$ into $L(S)$. Then, as $g \circ i$ is a $\mathbb{Z}_n$-valued measure on $L(\mathbb{R}^{2n})$, it follows from Lemma 2 that $(g \circ i)(\mathbb{R}^{2n}) = 0$, hence $g(S) = 0$. Suppose S is a finite-dimensional subspace of $\mathcal{H}$. Then $S^\perp$ and $\mathcal{H}$ are infinite-dimensional, giving $g(S^\perp) = 0 = g(\mathcal{H})$. But $g(S) + g(S^\perp) = g(\mathcal{H})$, and it follows that $g(S) = 0$.

Suppose $f : L(\mathcal{H}) \to G$ is a group-valued measure into a finite Abelian group G. As G is isomorphic to a direct sum of finite cyclic groups, an element $a \in G$ is equal to zero if, and only if, $h(a) = 0$ for every homomorphism $h : G \to \mathbb{Z}_n$ from G into a finite cyclic group. Suppose then that S is a closed subspace of $\mathcal{H}$ and that $h : G \to \mathbb{Z}_n$ is a homomorphism. Then $h \circ f : L(\mathcal{H}) \to \mathbb{Z}_n$ is a group-valued measure, hence $h \circ f$ is identically zero. It follows that $h(f(S)) = 0$ for each such homomorphism h, and therefore that $f(S) = 0$. □

In David Foulis' study of *algebraic measure theory* (Bennett and Foulis, 1997; Foulis *et al.*, 1998; Foulis, 2000), a type of generalized orthomodular structure L, known as an effect algebra, replaces a Boolean algebra as the basic measure carrying vehicle. One then attempts to study L and its measures by constructing a partially ordered Abelian group U with order unit u such that (i) L is isomorphic to the unit interval of U, (ii) U is generated as an Abelian group by its unit interval, and (iii) the group-valued measures on L are exactly the restrictions of group homomorphisms $\varphi : U \to G$ on U.

One cannot find such an ordered Abelian group U for every effect algebra L, but if such U can be found, it is unique up to isomorphism and is called the universal group, or unigroup, of L. It is known that if L is the orthomodular lattice of closed subspaces of a Hilbert space $\mathcal{H}$, then L has a unigroup, however no useful description of this unigroup is known, except in the case that $\mathcal{H}$ is dimension two. Below we provide a small step toward such a description in the case that $\mathcal{H}$ is of infinite dimension. Recall that an Abelian group G is called divisible if for each

$g \in G$ and each natural number n there is an element $h \in G$ with $nh = g$ (here $2h = h + h$, etc.).

Theorem 7. *If $\mathcal{H}$ is an infinite-dimensional Hilbert space, then the unigroup of $L(\mathcal{H})$ is a divisible group.*

Proof: It is well known that an Abelian group is divisible if, and only if, every homomorphism from it into a finite Abelian group is identically zero (Fuchs, 1958). Let U be the unigroup of $L(\mathcal{H})$ and suppose $f : U \rightarrow G$ is a homomorphism from U into a finite Abelian group G. Then the restriction of f to $L(\mathcal{H})$ is a group-valued measure, so by the Theorem 6 the restriction of f to $L(\mathcal{H})$ is identically zero. Then as $L(\mathcal{H})$ generates U, it follows that the homomorphism f must be identically zero. □

5. CONCLUDING REMARKS

Our results are the first steps toward determining the group-valued measures on the orthomodular lattice of closed subspaces of a Hilbert space. We believe a solution to the following question would yield further progress.

Question 1. *Are the constant function zero and the dimension function modulo two the only $\mathbb{Z}_2$-valued measures on $L(\mathbb{R}^3)$?*

Consideration of various proofs of Gleason's theorem and the Kochen–Specker theorem did not lead us immediately to a positive solution to Question 1. If one is interested in attempting to find a negative solution to this question, it may be worthwhile to make note of various results on coloring the rational rays in $\mathbb{R}^3$ (see Havlicek *et al.*, 2001 for access to the literature). These results provide a real-valued measure taking values 0 and 1 on the orthomodular lattice $L(\mathbb{Q}^3)$ of subspaces of $\mathbb{Q}^3$. Of course, this result is of no benefit in producing a real-valued measure taking values 0 and 1 on $L(\mathbb{R}^3)$ (Gleason's theorem, or the Kochen–Specker theorem, shows there are none!), but there is at least the possibility that it may help in producing a non-trivial $\mathbb{Z}_2$-valued measure on $L(\mathbb{R}^3)$.

Further work toward a characterization of the unigroup of the orthomodular lattice of closed subspaces of a Hilbert space would also be desirable. Here there is a natural question to consider.

Question 2. *If* $\dim \mathcal{H} \geq 3$ *is the unigroup of $L(\mathcal{H})$ naturally isomorphic to the subgroup of the group of self-adjoint operators generated by the projections?*

We note that a negative answer to the first question would imply a negative answer to the second for $L(\mathbb{R}^3)$. Indeed, a negative answer to the first would yield a $\mathbb{Z}_2$-valued measure on $L(\mathbb{R}^3)$ with $f(\langle a \rangle) = 0$ and $f(\langle b \rangle) = 1$ for some orthogonal vectors a, b. Using a unitary transformation if necessary, we can assume $a = (1, 0, 0)$ and $b = (0, 1, 0)$. Set $c = (1, \sqrt{3}, 0)$ and $d = (1, -\sqrt{3}, 0)$ and let A, B, C, D be the projections onto $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$ and $\langle d \rangle$, respectively. With some elementary linear algebra we see

$$A + 3B = 2C + 2D.$$

It follows that f cannot possibly be lifted to a homomorphism f^* from the subgroup of the group of self-adjoint operators generated by the projections to $\mathbb{Z}_2$ as any such homomorphism f^* would satisfy

$$f^*(A + 3B) = f(\langle a \rangle) + 3f(\langle b \rangle) = 1,$$

and

$$f^*(2C + 2D) = 2f^*(C) + 2f^*(D) = 0.$$

A positive solution to Question 2 seems plausible, and would provide a nice link between the self-adjoint operators on $\mathcal{H}$ and the orthomodular lattice $L(\mathcal{H})$.

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